Reconstructing Quantum States Via Unambiguous State Discrimination

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Abstract

In this paper, we introduce an analytical framework for reconstructing of the quantum states. The reconstruction of an unknown quantum state requires the information of a complete set of observables that are obtained through experimental measurements of Hermitian operators usually defined as positive-operator-valued measures (POVMs). The scheme involves the single-qubit unambiguous state discrimination (USD) POVM, which can be generalized to perform n-qubit measurements. We also use Maximum likelihood estimation as a method in the reconstruction of the density matrix from experimental data and show that the expected value of the cleaner is independent of the parameter of density operator.

Keywords: Quantum Tomography, State reconstruction, POVM, State Discrimination, Maximum likelihood estimation

1 Introduction

The state of a system in classical physics is determined by some numbers, and it is usually always possible to find measurements that can completely recover the state of the system. On the other hand, the entangled states are important application resources in many branches of quantum communication and computing [1, 2], such as quantum teleportation [3], superdense coding [4], quantum key distribution (QKD) [5], etc. Therefore, the ability to characterize entangled states based on measurements is of interest to many researchers. Two basic features of quantum mechanics, namely no-cloning theorem and Heisenberg uncertainty, have prevented this from being possible in the world of quantum mechanics. According to no-cloning theorem, it is forbidden to make the same copies of an arbitrary quantum state without already knowing its state in advance [6]. Beside this, Heisenberg's uncertainty principle states that even if the measuring device is ideal, due to the presence of a random component in the measurements, the measurement results give only limited information about the state of the system [7, 8].

Estimation theory is a branch of statistics that studies the effect of the randomness of measurements on estimation accuracy and estimates the values of parameters [9]. In this theory, two approaches are generally considered [10]. The probabilistic approach assumes that the measured data is random with probability distribution dependent on the parameters of interest. The set-membership approach assumes that the measured data vector belongs to a set that depends on the parameter vector. Quantum state tomography (QST), as an example of quantum estimation, is the process of the full description of the quantum state by the estimation of a set of parameters large enough[11, 12]. This field of research in quantum physics is of great theoretical and experimental importance. The maximum possible information on some entangled states of W-type with trapped ions has been obtained via state tomography [13]. Also, to characterize the quantum state of an optical entangling gate, the QST has been used [14]. The applications of machine learning in the various subfields of quantum information science, including QST, have been studied by many scientists [15, 16].

Novel tomography schemes have been developed that employ generative machine learning models, enabling quantum state reconstruction from limited classical data. In [17] a pipeline of machine learning models for quantum state estimation using projective measurements was built.

In this work, we introduce unambiguous state discrimination(USD) POVM as a set of measurements on a density operator of the system. Using Maximum likelihood estimation as a method in the reconstruction of the density matrix from experimental data we show that our method has minimum variance.

2 Reconstruction of the one-qubit quantum state

Consider an ensemble of qubits described by the following density matrix:

$$\rho = \frac{1}{2}(I + s.\sigma) = \frac{1}{2}(I + \sum_{i=1}^{3} s_i\sigma_i), \tag{1}$$

such that I, σ_i $(i = 1, ..., 3)$ are the identity and Pauli matrices, also s_x, s_y are known and s_z is unknown. Our goal is to determine the optimal value of s_z. To turn the problem into a state discrimination problem, we write this density operator as a convex combination of two known pure states:

$$\rho = p \mid n_1\rangle\langle n_1 \mid +(1-p) \mid n_2\rangle\langle n_2 \mid, \tag{2}$$

where p is an unknown parameter. Since these two states are pure, they must be placed on the Bloch sphere according to Fig.(1).

In Fig. (1), the vertical diameter of the Bloch sphere represents the z-axis,

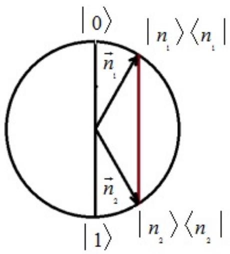

Fig. 1 Bloch sphere

and the points on the chord between the ends of the vectors $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$ show the state ρ for different values of p. By comparing Eq.(1) and Eq.(2), one can obtain $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$. To optimally estimate the density matrix ρ, must make measurements on the ensemble that optimally lead to obtain p. Measurement is defined on the system state with the set $\{M_k\}$ that M_k are measurement operators. These operators apply to the density matrix, so that the state of the system after the measurement is the following form:

$$\frac{M_k\rho M_k^\dagger}{Tr(M_k\rho M_k^\dagger)}, \tag{3}$$

which Tr represents the trace. The probability that we have the above output after the measurement is:

$$Tr(M_k\rho M_k^\dagger) = Tr(\rho M_k^\dagger M_k) = Tr(\Pi_k\rho). \tag{4}$$

The set Π_k introduces the operators that, according to the Born rule, give the probability of obtaining the result k. This means that:

$$p_k = Tr(\Pi_k \rho). \tag{5}$$

Π_k satisfy the following conditions:

$$\sum_k \Pi_k = I$$
$$\Pi_k \geq 0.$$

If M_ks are the projectors of the eigenvalues of an observable, then they are also orthogonal, i.e.:

$$M_k M_{k'} = \delta_{kk'} M_k.$$

In the following, we will show that the lower bound of the projective measurements satisfies Cramer-Rao inequality [18].
In the case of the single qubit problem mentioned earlier, to determine the optimal value of p, we introduce the following positive operators, which, although not orthogonal, completely distinguish the $\mid n_1\rangle$ and $\mid n_2\rangle$ states [19–22],

$$\Pi_1 = q \mid -n_2\rangle\langle -n_2 \mid,$$

$$\Pi_2 = q \mid -n_1\rangle\langle -n_1 \mid,$$
$$\Pi_3 = I - (q \mid -n_1\rangle\langle -n_1 \mid + q \mid -n_2\rangle\langle -n_2 \mid), \tag{6}$$

such that:

$$\sum_{i=1}^{3} \Pi_i = I.$$

Also, the effect of each of these operators on the pure ensemble $\mid n_2\rangle\langle n_2 \mid$ is:

$$Tr(\Pi_1 \mid n_2\rangle\langle n_2 \mid) = Tr(q \mid -n_2\rangle\langle -n_2 \mid\mid n_2\rangle\langle n_2 \mid) = 0$$

$$Tr(\Pi_2 \mid n_2\rangle\langle n_2 \mid) = Tr(q \mid -n_1\rangle\langle -n_1 \mid\mid n_2\rangle\langle n_2 \mid) = q \mid \langle -n_1 \mid n_2\rangle \mid^2$$
$$Tr(\Pi_3 \mid n_2\rangle\langle n_2 \mid) = Tr\{(I - \Pi_1 - \Pi_2) \mid n_2\rangle\langle n_2 \mid\} = 1 - q \mid \langle -n_1 \mid n_2\rangle \mid^2 .$$

The above equations state that the probability of the outcome of the pure ensemble $\mid n_2\rangle\langle n_2 \mid$ from output 1 is zero, while this probability for the output mentioned ensemble from output 2 is equal to $q \mid \langle -n_2 \mid n_1\rangle \mid^2$. Also, for output 3 this probability is non-zero. So, there are two states for the pure ensemble $\mid n_2\rangle\langle n_2 \mid$: With the probability $q \mid \langle -n_2 \mid n_1\rangle \mid^2$ exists from the output 2 or with the probability $1 - q \mid \langle -n_2 \mid n_1\rangle \mid^2$ exists from the output 3. We know that if the state exits from output 2, the qubit state will be exactly $\mid n_2\rangle$, and if it exits from output 3, the system state cannot be judged. The result is that the third output does not provide any information about the exact type of output state. Therefore, the information in this output is not usable.
We can provide a similar analysis in a similar way for the pure ensemble

$\mid n_1\rangle\langle n_1 \mid$. We consider the following relations:

$$Tr(\Pi_1 \mid n_1\rangle\langle n_1 \mid) = Tr(q \mid -n_2\rangle\langle -n_2 \mid\mid n_1\rangle\langle n_1 \mid) = q \mid \langle -n_2 \mid n_1\rangle \mid^2$$
$$Tr(\Pi_2 \mid n_1\rangle\langle n_1 \mid) = Tr(q \mid -n_1\rangle\langle -n_1 \mid\mid n_1\rangle\langle n_1 \mid) = 0$$
$$Tr(\Pi_3 \mid n_1\rangle\langle n_1 \mid) = Tr\{(I - \Pi_1 - \Pi_2) \mid n_1\rangle\langle n_1 \mid\} = 1 - q \mid \langle -n_2 \mid n_1\rangle \mid^2 .$$

In a similar way, the probability of the ensemble exiting $\mid n_1\rangle\langle n_1 \mid$ from output 2 is zero and from outputs 1 and 3 is non-zero. If the ensemble exits from output 1, the state of the ensemble is certainly $\mid n_1\rangle$; otherwise, the ensemble exits from output 3, which does not provide information about the ensemble type. Without loss of generality of the problem, we consider output 3 as lost information and ignore its results. Therefore, the result of measurement either determines exactly the type of state or the information is considered unusable. The above issues can be shown schematically in the following figures. In total there are two possibilities which are shown in the following figures:
Fig.(2) shows that the state with a possible percentage is out of output 1, in

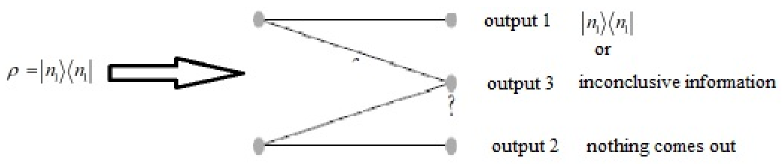

Fig. 2 Cleansing channel for the pure ensemble $\mid n_1\rangle\langle n_1 \mid$

which case we will definitely have no state in output 2. In other words, a pure ensemble $\mid n_1\rangle\langle n_1 \mid$ either exits from the output 1 or exits from output 3 as inconclusive information at most and never exits particle from output 2 in any way. Similarly, it can be shown in Fig.(3) that if a pure ensemble $\mid n_2\rangle\langle n_2 \mid$ enters the channel, it either exits from output 2, or exits from output 3 as inconclusive information, and no particle will pass through output 1. Similarly, for the density operator $\rho = p \mid n_1\rangle\langle n_1 \mid +(1-p) \mid n_2\rangle\langle n_2 \mid$, the cleaning channel can be shown in Fig.(4).

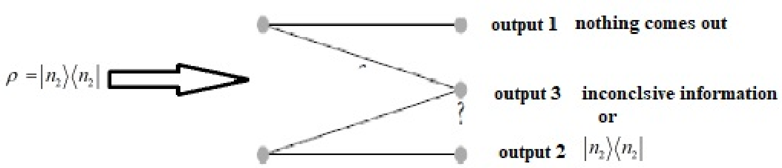

Fig. 3 Cleansing channel for the pure ensemble $\mid n_2\rangle\langle n_2 \mid$

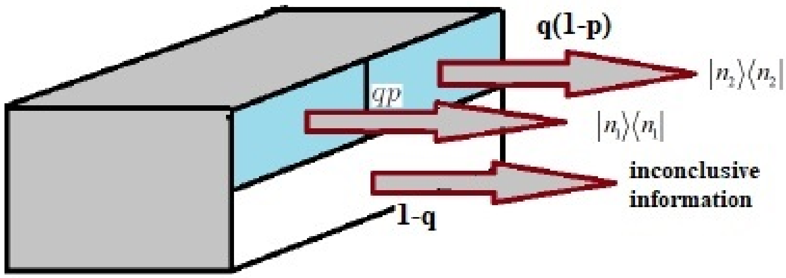

Fig. 4 Cleansing channel for the density operator ρ

The important note is that we should reduce the contribution of Π_3 as much as possible. Since the larger the value of the q, the smaller the contribution of the cleansing channel, we should make the q as large as possible so that the eigenvalues of Π_3 remain positive. For this purpose, we choose q in such a way that the smallest eigenvalue of the Π_3 is equal to zero.
We have already introduced the operator as follows:

$$\Pi_3 = I - \Pi_1 - \Pi_2 = I - q(| -n_1\rangle\langle -n_1 \mid + \mid -n_2\rangle\langle -n_2 \mid). \tag{7}$$

Since the identity operator is commutative with any other operator, it is sufficient to find the eigenvalues of the phrase in the parentheses.

$$| -n_1\rangle\langle -n_1 \mid + \mid -n_2\rangle\langle -n_2 \mid = \frac{1}{2}(I - n_1.\sigma) + \frac{1}{2}(I - n_2.\sigma)$$

$$= I - \frac{(n_1 + n_2).\sigma}{2}. \tag{8}$$

$$(n_1 + n_2).\sigma = \begin{pmatrix} n_{1z} + n_{2z} & n_{1x} + n_{2x} - i(n_{1y} + n_{2y}) \\ n_{1x} + n_{2x} + i(n_{1y} + n_{2y}) & -n_{1z} - n_{2z} \end{pmatrix}. \tag{9}$$

The eigenvalues of the above operator are:

$$\lambda_{\pm} = \pm\sqrt{(n_{1x} + n_{2x})^2 + (n_{1y} + n_{2y})^2 + (n_{1z} + n_{2z})^2} = \pm \mid n_1 + n_2 \mid, \tag{10}$$

which $|$ ... $|$ indicates the Euclidean norm. Finally, the eigenvalues of Π_3 are:

$$\Lambda_{\pm} = 1 - q(1 \pm \frac{\mid n_1 + n_2 \mid}{2}), n_1 = \begin{pmatrix} s_x \\ s_y \\ s_z \end{pmatrix}, n_2 = \begin{pmatrix} s_x \\ s_y \\ -s_z \end{pmatrix} \tag{11}$$

$$\Lambda_{\pm} = 1 - q(1 \pm \sqrt{s_x^2 + s_y^2}). \tag{12}$$

We make zero the small eigenvalue:

$$1 - q(1 + \frac{\mid n_1 + n_2 \mid}{2}) = 0 \Rightarrow q = \frac{1}{1 + \frac{|n_1+n_2|}{2}} \tag{13}$$

$$q = \frac{1}{1+\sqrt{s_x^2+s_y^2}}. \tag{14}$$

Given that the q has a value between zero and one, so the above value is the maximum value that the q can take. As a result, the q range can be expressed as follows:

$$0 \leq q \leq \frac{1}{1+\sqrt{s_x^2+s_y^2}}. \tag{15}$$

By calculating the probabilities associated with each of the outputs, we can determine the probability of exiting from the cleaner.

$$Tr(\Pi_1\rho) = (q \mid -n_2\rangle\langle -n_2 \mid)(p \mid n_1\rangle\langle n_1 \mid +(1-p) \mid n_2\rangle\langle n_2 \mid),$$

$$\begin{gathered} Tr(\Pi_2\rho) = (q \mid -n_1\rangle\langle -n_1 \mid)(p \mid n_1\rangle\langle n_1 \mid +(1-p) \mid n_2\rangle\langle n_2 \mid), \\ Tr(\Pi_3\rho) = 1 - Tr(\Pi_1\rho) - Tr(\Pi_2\rho), \end{gathered} \tag{16}$$

so

$$\begin{gathered} p_1 = Tr(\Pi_1\rho) = qp \mid \langle -n_2 \mid n_1\rangle \mid^2, \\ p_2 = Tr(\Pi_2\rho) = q(1-p) \mid \langle -n_1 \mid n_2\rangle \mid^2, \\ p_3 = Tr(\Pi_3\rho) = 1 - (qp \mid \langle -n_2 \mid n_1\rangle \mid^2 + q(1-p) \mid \langle -n_1 \mid n_2\rangle \mid^2). \end{gathered} \tag{17}$$

To calculate the inner products that appear in the above statements, we use the following method:

$$\mid -n_2\rangle\langle -n_2 \mid = \frac{1}{2}(I - n_2.\sigma),$$

$$\begin{gathered} \mid n_1\rangle\langle n_1 \mid = \frac{1}{2}(I + n_1.\sigma), \\ \mid \langle -n_2 \mid n_1\rangle \mid^2 = Tr(\mid -n_2\rangle\langle -n_2 \mid n_1\rangle\langle n_1 \mid) \\ = Tr\{\frac{1}{2}(I - n_2.\sigma)\frac{1}{2}(I + n_1.\sigma)\} \\ = Tr(\frac{I + n_1.\sigma - n_2.\sigma - (n_1.\sigma)(n_2.\sigma)}{4}) \\ = \frac{2 - Tr((n_1.\sigma)(n_2.\sigma))}{4} = \frac{2 - Tr(n_1.n_2 I + i(n_1 \times n_2).\sigma)}{4} = \frac{1 - n_1.n_2}{2}. \end{gathered}$$

Similarly, we have:

$$\mid \langle -n_1 \mid n_2\rangle \mid^2 = \frac{1 - n_1.n_2}{2}.$$

For the two states with the following Bloch vectors:

$$n_1 = \begin{pmatrix} s_x \\ s_y \\ s_z \end{pmatrix}, n_2 = \begin{pmatrix} s_x \\ s_y \\ -s_z \end{pmatrix}$$

$$\frac{1-n_1.n_2}{2}=\frac{1-s_x^2-s_y^2+s_z^2}{2}$$
$$=\frac{1-s_x^2-s_y^2+1-s_x^2-s_y^2}{2}=1-s_x^2-s_y^2. \quad (18)$$

By replacing the recent result in the Eqs.(17) we have:

$$p_1=Tr(\Pi_1\rho)=qp(1-s_x^2-s_y^2)$$

$$p_2=Tr(\Pi_2\rho)=q(1-p)(1-s_x^2-s_y^2)$$
$$p_3=Tr(\Pi_3\rho)=1-q(1-s_x^2-s_y^2). \quad (19)$$

It is observed that the expected value of the cleaner is not dependent on the unknown parameter p. This interesting result indicates that this value does not depend on the selected pure ensembles and only depends on the selected measurement set. Therefore, we leave out the inconclusive information coming out of the cleaner, and count only the particles that come out of outputs 1 and 2. To simplify the calculations, we can normalize the above probabilities between outputs 1 and 2 by dividing the constant coefficient displayed by the probability values 1 and 2. Finally, we will see that this coefficient will not be included in estimating the value of the p-parameter using the maximum likelihood estimation method.

3 Maximum estimation method in the parameter estimating

As mentioned earlier, tomography is the reconstruction of an ensemble state of quantum particles that are all prepared in the same state. So any number can be measured on this ensemble. Because the measurement is a random phenomenon, there is an uncertainty due to the randomness of the measurement outputs. Therefore, if we want to reconstruct the density operator from this experimental data, the related reconstruction process is not accurate and is always accompanied by an error. As a result, with a statistical error, we can obtain the probabilities corresponding to each of the measurement results from the experimental data. The parameter λ, for which the density operator ρ_λ is expressed, is determined from the measurement data so that the measurement output probabilities are optimally close to the probabilities obtained from the experimental data. For this purpose, we use the maximum likelihood estimation method. In this method, we maximize the distribution function of the test outputs relative to the parameter λ. We denote the mentioned distribution function by $f(X \mid \lambda)$. X is a random variable that represents a set of possible measured outputs. X and λ can be vectors.
For single qubit and related density operators, the output probabilities were described using the measurement operators Π_i. If we divide these relations by

a constant coefficient $q(1-s_x^2-s_y^2)$, we have:

$$\widetilde{p_1}=\frac{p_1}{q(1-s_x^2-s_y^2)}=p,$$

$$\widetilde{p_2}=\frac{p_2}{q(1-s_x^2-s_y^2)}=1-p,$$
$$\widetilde{p_3}=\frac{p_3}{q(1-s_x^2-s_y^2)}. \tag{20}$$

Since $\widetilde{p_3}$ does not depend on the parameter p, we exclude the corresponding measurement outputs as a laboratory error. The advantage of the working with $\widetilde{p_1}$ and $\widetilde{p_2}$ is that their sum is normalized to one. If we take the Bernoulli distribution of the measurement distribution function, we have:

$$f(X\mid\widetilde{p})=\widetilde{p_1}^{x_1}\widetilde{p_2}^{x_2}=p^{x_1}(1-p)^{x_2},$$

$$x_1,x_2\in\{0,1\},x_1+x_2=1. \tag{21}$$

The likelihood function L for a sample obtained from the data of the N independent tests is equal to the product of the likelihood function of each experiment. We represent the sample elements related to the random variable x_i with x_{ik}.

$$L=\prod_{k=1}^{N}p^{x_{1k}}(1-p)^{x_{2k}},$$
$$L=p^{\sum_{k=1}^{N}x_{1k}}(1-p)^{\sum_{k=1}^{N}x_{2k}}, \tag{22}$$
$$x_{ik}\in\{0,1\},\sum_{i=1}^{2}x_{ik}=1.$$

Since the likelihood function is a positive function, the maximum of this function and its logarithm occur at a common point. By taking the logarithm from the sides, we have:

$$logL=(\sum_{k=1}^{N}x_{1k})log(p)+(\sum_{k=1}^{N}x_{2k})log(1-p). \tag{23}$$

By deriving respect to the parameter p and making it zero, we have:

$$\frac{\partial logL}{\partial p}=\frac{\sum_{k=1}^{N}x_{1k}}{p}+\frac{\sum_{k=1}^{N}x_{2k}}{1-p}=0.$$

If we denote the number of particles exited by output 1 by n_1 and the number of particles exited by output 2 by n_2, such that the sum of the number of particles is equal to the constant value of N, then we have:

$$\begin{aligned} \sum_{k=1}^{N} x_{1k} &= n_1, \\ \sum_{k=1}^{N} x_{2k} &= n_2, \\ n_1 + n_2 &= N, \\ p = \frac{\sum_{k=1}^{N} x_{1k}}{N} &= \frac{n_1}{N}. \end{aligned} \tag{24}$$

On the other hand, the recent relation is the mean of the random variable x_1, i.e:

$$p = E(x_1) = \frac{\sum_{k=1}^{N} x_{1k}}{N}, \tag{25}$$

in which $E(x_1)$ denotes the mean of the x_1.

4 Estimation error

In this section, we calculate the error related to estimating the parameter p and show that this error is minimal according to the Cramer-Rao bound. One measure of the data sparsity around the mean value is variance. Next, we calculate the variance of the data and show that the result is equal to the lower bound of the Cramer-Rao inequality.

4.1 Calculation of the estimation error from the measurement data

We calculate the variance of the estimated parameter p as follows.

$$\begin{aligned} Var(p) &= Var(\frac{\sum_{k=1}^{N} x_{1k}}{N}) \\ &= \frac{1}{N^2} Var(\sum_{k=1}^{N} x_{1k}) = \frac{1}{N^2} \sum_{k=1}^{N} Var(x_{1k}) \\ &= \frac{1}{N^2} \sum_{k=1}^{N} (x_{1k} - E(x_1))^2 \\ &= \frac{1}{N^2} (n_1 (1 - E(x_1))^2 + (N - n_1)(0 - E(x_1))^2) \\ &= \frac{1}{N} (\frac{n_1}{N} (1-p)^2 + (1 - \frac{n_1}{N})(0-p)^2) \\ &= \frac{1}{N} (p(1-p)^2 + (1-p)p^2) = \frac{1}{N} p(1-p). \end{aligned} \tag{26}$$

We obtain this result again using the distribution function $f(X \mid \widetilde{p})$ as follows:

$$\begin{aligned} Var(p) = \frac{Var(x_1)}{N} &= \frac{1}{N} \sum_{x_1+x_2=1} (x_1 - E(x_1))^2 f(X \mid \widetilde{p}) \\ &= \frac{1}{N} \sum_{x_1+x_2=1} (x_1 - p)^2 p^{x_1} (1-p)^{x_2} \\ &= \frac{1}{N}((1-p)^2 p^1 (1-p)^0 + (0-p)^2 p^0 (1-p)^1 \\ &= \frac{1}{N}((1-p)^2 p + p^2(1-p)) = \frac{1}{N} p(1-p). \end{aligned} \tag{27}$$

It can be seen that the same result of Eq.(26) was obtained correctly. Note that the sum was performed on the values that satisfy the following equation:

$$x_1 + x_2 = 1, \qquad x_1, x_2 \in \{0, 1\}. \tag{28}$$

We used the point that the above equation is a fluid equation and has only two answers as follows on the set $\{0, 1\}$:

$$\begin{aligned} x_1 &= 0, x_2 = 1 \\ x_1 &= 1, x_2 = 0. \end{aligned} \tag{29}$$

5 Calculation of Classical Fisher information to estimate p

We saw that in estimating the unknown parameter, we always encounter an unavoidable error. This error is due to the random nature of the phenomenon under study. At best, our estimates are scattered around the actual value of the estimated parameter. We expressed a measure of this scatter by variance. The amount of this variance can never be less than a certain amount. So there is a lower bound to the estimated variance that cannot be less than that. The most famous lower bound for the variance of an estimate is the Cramer-Rao lower bound, which is expressed as follows:

$$Var(\lambda) \geq \frac{1}{NF(\lambda)}, \tag{30}$$

that $F(\lambda)$ is called Classical Fisher Information and N is the number of samples or the number of measurements. We want to show that the variance we obtain for the unknown parameter p is equal to the lower bound of the Cramer-Rao inequality. If we have only one parameter and the desired random variable is continuous, $F(\lambda)$ is defined by the following integral:

$$F(\lambda) = \int dx p(x \mid \lambda) (\frac{\partial ln p(x \mid \lambda)}{\partial \lambda})^2$$

$$= \int dx \frac{1}{p(x \mid \lambda)} (\frac{\partial p(x \mid \lambda)}{\partial \lambda})^2, \tag{31}$$

that $p(x \mid \lambda)$ is the conditional probability of obtaining x when the value of the parameter is λ. If the random variable X is discrete, the above definition is as follows:

$$F(\lambda) = \sum_X p(X \mid \lambda)(\frac{\partial lnp(X \mid \lambda)}{\partial \lambda})^2$$
$$= \sum_X \frac{1}{p(X \mid \lambda)} (\frac{\partial p(X \mid \lambda)}{\partial \lambda})^2. \tag{32}$$

Now, we get the classical Fisher information corresponding to p. we have:

$$F(p) = \sum_X \frac{1}{f(X \mid \widetilde{P})} (\frac{\partial f(X \mid \widetilde{P})}{\partial \lambda})^2, \tag{33}$$

which we get using the Eq.(21):

$$F(p) = \sum_{x_1+x_2=1} \frac{1}{p^{x_1}(1-p)^{x_2}} (x_1 p^{x_1-1}(1-p)^{x_2} - x_2 p^{x_1}(1-p)^{x_2-1})^2$$

$$= \frac{1}{p} + \frac{1}{1-p} = \frac{1}{p(1-p)}. \tag{34}$$

With this result, we can write inequality Eq.(30) as follows:

$$Var(p) \geq \frac{1}{NF(p)} = \frac{p(1-p)}{N}. \tag{35}$$

On the other hand, we obtained the following value for variance p according to Eq.(27):

$$Var(p) = \frac{p(1-p)}{N}. \tag{36}$$

By comparing these two relations, we find that:

$$Var(p) = \frac{1}{NF(p)}. \tag{37}$$

Therefore, our estimate gives the lowest classical variance allowed.

6 Conclusion

In summary, we have investigated the single-qbit state estimation using USD bases which is applicable to higher dimensions. Also, the maximum likelihood estimation as a method in reconstruction density matrix of single qubit from experimental data have been used. We observed that the expected value of the cleaner is not dependent on the unknown parameter p. Hopefully these line of research will be pursued further for N-qubit measurement.

7 Research Data Policy and Data Availability Statements

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

8 Acknowledgments

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References

[1] Einstein, A., Podolsky, B., Rosen, N.: Can quantum-mechanical description of physical reality be considered complete? Physical review **47**(10), 777 (1935)

[2] Bell, J.S.: On the einstein podolsky rosen paradox. Physics Physique Fizika **1**(3), 195 (1964)

[3] Anderson, M., Müller, T., Skiba-Szymanska, J., Krysa, A., Huwer, J., Stevenson, R., Heffernan, J., Ritchie, D., Shields, A.: Gigahertz-clocked teleportation of time-bin qubits with a quantum dot in the telecommunication c band. Physical Review Applied **13**(5), 054052 (2020)

[4] Bennett, C.H., Wiesner, S.J.: Communication via one-and two-particle operators on einstein-podolsky-rosen states. Physical review letters **69**(20), 2881 (1992)

[5] Ekert, A.K.: Quantum cryptography and bell's theorem. In: Quantum Measurements in Optics, pp. 413–418. Springer, Boston (1992)

[6] Helstrom, C.W.: Quantum detection and estimation theory. Journal of Statistical Physics **1**(2), 231–252 (1969)

[7] Nielsen, M.A., Chuang, I.: Quantum computation and quantum information. American Association of Physics Teachers (2002)

[8] Peres, A.: Quantum Theory: Concepts and Methods. Springer, Dordrecht: Kluwer Academic (2002)

[9] Kay, S.M.: Fundamentals of Statistical Signal Processing: Estimation Theory. Prentice-Hall, Inc., Prentice-Hall (1993)

[10] Walter, E., Pronzato, L., Norton, J.: Identification of Parametric Models from Experimental Data vol. 1. Springer, Berlin (1997)

[11] Liu, Y.-x., Wei, L., Nori, F.: Tomographic measurements on superconducting qubit states. Physical Review B **72**(1), 014547 (2005)

[12] D'Ariano, G.M., Paris, M.G., Sacchi, M.F.: Quantum tomography. Advances in Imaging and Electron Physics **128**, 206–309 (2003)

[13] Häffner, H., Hänsel, W., Roos, C., Benhelm, J., Chek-al-Kar, D., Chwalla, M., Körber, T., Rapol, U., Riebe, M., Schmidt, P., *et al.*: Scalable multiparticle entanglement of trapped ions. Nature **438**(7068), 643–646 (2005)

[14] O'Brien, J.L., Pryde, G.J., White, A.G., Ralph, T.C., Branning, D.: Demonstration of an all-optical quantum controlled-not gate. Nature **426**(6964), 264–267 (2003)

[15] Lohani, S., Kirby, B.T., Brodsky, M., Danaci, O., Glasser, R.T.: Machine learning assisted quantum state estimation. Machine Learning: Science and Technology **1**(3), 035007 (2020)

[16] Zimmermann, B., Mueller, T., Meineke, J., Esslinger, T., Moritz, H.: High-resolution imaging of ultracold fermions in microscopically tailored optical potentials. New Journal of Physics **13**(4), 043007 (2011)

[17] Danaci, O., Lohani, S., Kirby, B.T., Glasser, R.T.: Machine learning pipeline for quantum state estimation with incomplete measurements. Machine Learning: Science and Technology **2**(3), 035014 (2021)

[18] Kullback, S.: Certain inequalities in information theory and the cramer-rao inequality. The Annals of Mathematical Statistics, 745–751 (1954)

[19] Karimi, N.: Optimal unambiguous discrimination of pure quantum states using sdp method. Chinese Journal of Physics **72**, 681–687 (2021)

[20] Jafarizadeh, M., Rezaei, M., Karimi, N., Amiri, A.: Optimal unambiguous discrimination of quantum states. Physical Review A **77**(4), 042314 (2008)

[21] Pang, S., Wu, S., *et al.*: Optimum unambiguous discrimination of linearly independent pure states. Physical Review A **80**(5), 052320 (2009)

[22] Bergou, J.A., Futschik, U., Feldman, E.: Optimal unambiguous discrimination of pure quantum states. Physical review letters **108**(25), 250502 (2012)