VALUATION RING

1. DEFINITIONS AND PRELIMINARIES

Definition 1.1. *Let R be an integral domain, K its field of fractions. R is a valuation ring of K if, for each $o \neq x \in K$, either $x \in R$ or $x^{-1} \in R$ (or both).*

$[M.F.Atiyah\ and\ I.G.Macdonald, Introduction\ to\ Commutative\ Algebra, page 65]$

Note
Suppose that K is a field and R a subring of K such that above condition is true, then K is fraction field of R. So always we consider K as fraction field of R

Example

(1) Any field is a valuation rings.
(2) The ring $\mathbb{Z}_{(2)}$ is a valuation ring of $\mathbb{Q}$. $\mathbb{Z}_{(2)} = \{\frac{a}{b}|(b,2) = 1\}$ is a subring of $\mathbb{Q}$. Let $\frac{m}{n} = x \in \mathbb{Q}, (m,n) = 1$. Then either m or n is odd, so $x \in \mathbb{Q}$ or $x^{-1} \in \mathbb{Q}$
(3) Let $K = K(x)$ and R be the set of all rational functions $\frac{f}{g} \in K(x)$ such that $deg\ f \leqslant deg\ g$. Then R is a valuation ring of K.

$[Pete\ L.Clark, Commutative\ Algebra, page 261]$

Definition 1.2. *Let K be a field and (A, m_A) , (B, m_B) be local rings contained in K. We say that B dominate A, if $A \subseteq B$, $m_A = A \cap m_B$*

$[N.Bourbaki, Commutative\ Algebra, page 375]$

Theorem 1.3. *Let R be a ring and K be fraction field of R. The following statements are equvalent:*

(1) *R is a valuation ring.*
(2) *for all $a, b \in R$, $a|b$ or $b|a$.(If I, J are any two ideal of R, Then either $I \subseteq J$ or $J \subseteq I$)*
(3) *R is a local domain and R is maximal for the relation of domination among local subrings of K.*
(4) *The set of principal ideals of A is totally ordered by the relation of inclusion.*
(5) *the set of ideals of A is totally ordered by the relation of inclusion.*

$[N.Bourbaki, Commutative\ algebra, page 375]$

Lemma 1.4. *Let R be a subring of a field K, and let $P \in SpecR$. Then there exists a valuation ring T of K such that $R \subseteq T$, $m_T \cap R = P$*

$[Pete\ L.Clark, Commutative\ Algebra, page 267]$

Corollary 1.5. *Let K be a field and $A \subseteq K$ be a local subring. Then there exists a valuation ring with fraction field K dominating A.*

$[N.Bourbaki, Commutative\ algebra, page 378]$

Lemma 1.6. *Let R be a valuation ring. For any prime ideal P in R, the quotient ring $\frac{R}{P}$ and localization R_P are valuation rings.*

Lemma 1.6 is not nessesery true for any ideal of R. For example $\mathbb{Z}_{(2)}$ is a valuation ring and its ideals are $I_2 \supset I_4 \supset I_8 \supset \dots$,where

$$I_k = \{\frac{m}{n} \mid k|m, (2,n) = 1\}.$$

I_4 is not prime and quotient ring $\frac{\mathbb{Z}_{(2)}}{I_4}$ is isomorphic to $\mathbb{Z}_4$ that is not a valuation ring.

Definition 1.7. *Let $A \subseteq B$ be two rings. An element $x \in B$ is integral over A if it is a root of a monic polynomial in $A[x]$*

The set C of elements of B which are integral over A is called the integral closure of A in B. If $C = A$, then A is said to be integrally closed in B.

Corollary 1.8. *Let R be a subring in a field K. Then the intersection of all valuation rings V of K which containes R is precisely the integral closure $\tilde{R}$ of R*

[M. F. Atiyah and I. G. Macdonald,
Introduction to Commutative Algebra, page 66]

Corollary 1.9. *Let R be a local subring in a field K. Then the intersection of all valuation rings V of K which dominating R is precisely the integral closure $\tilde{R}$ of R*

$[H.Matsumura, Commutative\ Ring\ Theory, page 77]$

Definition 1.10. *A totally ordered abelian group is a pair $(G, \leq)$, consisting of an abelian group G endowed with a total ordering $\leq$ such that $a+c \leq b+d$ when $a \leq b$, $c \leq d$ for all $a, b, c, d \in G$*

$[Pete\ L.Clark, Commutative\ Algebra, page 263]$

Definition 1.11. *Let $(G, \leq)$ be a totally ordered abelian group. A subgroup H of G is called an isolated subgroup of G if whenever $o \leqslant \beta \leqslant \alpha$ and $\alpha \in H$ we have $\beta \in H$.*

Example Let A and B be two ordered groups; let $A \times B$ be given the lexicographic order [1] . The second factor B (i.e.,$0 \times B$) of $A \times B$ is then an isolated subgroup of $A \times B$.

Definition 1.12. *A valuation on a field K is a function $v : K^* \longrightarrow G$ where G is a totally ordered abelian group such that,*

(1) $v(ab) = v(a) + v(b)$
(2) $v(a + b) \geq \min\{v(a), v(b)\}$

the subgroup $v(K^*)$ of G is called the value group of v.

$[J.J.Rotman, Advanced\ modern\ Algebra, page, 920]$

Proposition 1.13. *Let $v : K^* \longrightarrow G$ be a valuation on a field K and $R = \{a \in K^* | v(a) \geq o\} \cup \{o\}$. R is a valuation ring and every valuation ring arises in this way from a suitable valuation on its fraction field. Maximal ideal of R is $m = \{x \in K | v(x) > o$ or $x = o\}$ and its group of units is $A^* = \{x \in K^* | v(x) = o\}$.*

[1] Given two partially ordered sets A and B, the lexicographical order on the Cartesian product $A \times B$ is defined as $(a, b) \leq (c, d)$ if and only if $a < c$ or ($a = c$ and $b \leq d$).

$[H.Matsumura, Commutative\ Ring\ Theory, page75]$,
$[J.J.Rotman, Advanced\ modern\ Algebra, page, 920]$

Let A be a valuation ring of a field K. The group A^* of units of A is a subgroup of the multiplicative group K^* of K. Let $G = \frac{K^*}{A^*}$ and for $x, y \in K$ define $\bar{x} \geqslant \bar{y}$ to mean $xy^{-1} \in A$. This is a total ordering on G which is compatible with the group structure. So G is a totally ordered abelian group and it is called the value group of A. This is assign to valuation ring A a valuation with canonical homomorphism $\nu : K^* \longrightarrow G$. By the first isomorphism theorem, $\frac{K^*}{Ker(\nu)} \cong \nu(k^*) = G$. So $Ker(\nu) = A^*$

Thus the concept of valuation ring and valuation are essentially equivalent.

Example

(1) Let $K = \mathbb{Q}$ and p be a prime number of $\mathbb{Z}$. Then any nonzero element $x \in \mathbb{Q}$ can be written uniquely in the form $p^a y$, where $a \in \mathbb{Z}$ and $y \in \mathbb{Q}$ and both numerator and denumerator of y are prime to p. Define $\nu(x) = a$. Then the valuation ring of ν is $R = \mathbb{Z}_{(p)} = \{\frac{c}{d} | c, d \in \mathbb{Z}, p \nmid d\}$

(2) Let $K = F(x)$ and $f(x)$ be an irreducible polynomial of $F[x]$, where F is a field and x is an indeterminate over F. Then any nonzero polynomial $h(x) \in K$ can be written uniquely in the form $f(x)^a \frac{g(x)}{l(x)}$, where $a \in \mathbb{Z}$ and both $g(x)$, $l(x)$ are prime to $f(x)$. Define $\nu(h(x)) = a$. Then the valuation ring of ν is the local ring $F[x]_{(f)} = \{\frac{p(x)}{q(x)} | \ p(x), q(x) \in F[x], f(x) \nmid q(x)\}$.

$[J.J.Rotman, Advanced\ modern\ Algebra, page, 920]$

Question If R is a valuation ring , by 1.6 the localization R_P and qoutient ring $\dfrac{R}{P}$ are valuation rings. What are the value groups of R_P and qoutient ring $\dfrac{R}{P}$?

I think, Value group $\frac{R}{P}$ is an isolated subgroup of value group R and value group R_p is the same value group of R.

Theorem 1.14. *(Malcev, Neumann)*
For any ordered abelian group G there exists a valuation domain with value group isomorphic to G.

$[Pete\ L.Clark, Commutative\ Algebra, page265]$

Note If R is a valuation domain with value group G, there is a one to one order preserving correspondence between nonzero prime ideals of R and isolated proper subgroups of G. If $P \in Spec(R)$, $v(R - P)$ is the corresponding isolated subgroup of G .

Definition 1.15. *A discrete valuation ring (DVR) is a local PID that is not a field.*

Note The valuation ring is discrete when the totally ordered abelian group G is isomorphic to $\mathbb{Z}$.

Example

(1) As a simple example $\mathbb{Z}_{(P)}$ is a DVR.

(2) Another important example of a DVR is the ring of formal power series $R = K[[x]]$ in one variable x over some field K. The "unique" irreducible [2] element is x, the maximal ideal of R is the principal ideal generated by x, and the valuation assigns to each power series, the index (i.e. degree) of the first non-zero coefficient.
But for example $\mathbb{R}[x]$ is not DVR. since for all $a \in \mathbb{R}$, $< x - a >$ is a maximal ideal of $\mathbb{R}[x]$, that is not a local ring.

(3) $\mathbb{R}[x]_{<x-2>}$ is a DVR with maximal ideal $< x - 2 >$ and fraction field $\mathbb{R}(x)$.

Let R be a DVR , then any irreducible element of R is a generator for the unique maximal ideal of R and vice versa. Such an element is also called a a uniformizing element of R, a uniformizer, or a prime element. If we fix a uniformizing parameter x, then $m = (x)$ is the unique maximal ideal of R, and every other non-zero ideal is a power of m, i.e. has the form (x^k) for some $k \geq 0$. All the powers of x are distinct, and so are the powers of m. Every non-zero element r of R can be written in the form ax^k with a a unit in R and $k \geq 0$, both uniquely determined by r. The valuation is given by $\nu(r) = k$.

Proposition 1.16. *Let R be a DVR and (P) its maximal ideal. Then*

$$\bigcap_n (P^n) = 0.$$

$[H.A.Nielsen, Elementary\ Commutative\ Algebra, page\ 141]$

Theorem 1.17. *Let R be a local domain, P its non-zero maximal ideal,and P is principal, such that $\bigcap_n (P^n) = 0$. Then R is a DVR.*

$[A.Attman, S.Kleiman, A\ Term\ of\ Commutative\ Algebra, page\ 127]$

2. PROPERTIES OF VALUATION RING

Proposition 2.1. *Let R be a valuation ring,*

(1) *R is a local ring.*
(2) *If $\acute{R}$ is a ring such that $R \subseteq \acute{R} \subseteq K$, then $\acute{R}$ is a valuation ring of K.*
(3) *R is integrally closed in K.*

[M. F. Atiyah and I. G. Macdonald,
Introduction to Commutative Algebra, page 65]

Question

Let R be a disceret valuation ring, and S any subring of it. Is the subring S essentially a DVR?

Counter example

Let K be a field. The subring $R = K[[x^2, x^3]] \subseteq K[[x]]$ is not a disceret valuation ring. because it is not integrally closed. $x \notin R$ and is integral over R with integral dependence equation $x^2 t - x^3$.

[2] a non-zero non-unit element in an integral domain is said to be irreducible if it is not a product of two non-units.

Proposition 2.2. *If V is a valuation ring, then*

(1) *Every finitely generated ideal in V is principal.*
(2) *Every finitely generated ideal in V is projective.*

$[J.J.Rotman, Advanced\ modern\ Algebra, page 920]$

Definition 2.3. *A domain in which every finitely generated ideal is principal called a Bezout domain.*

[K. R. Goodeari and R. B. Warfield,
An Introduction to Noncommutative Noethrian Rings, page 112]

Proposition 2.4. *A local Bezout domain is a valuation ring.*

$[Pete\ L.Clark, Commutative\ Algebra, page 262]$

Proposition 2.5. *Every valuation ring is a Bezout domain. In particular, every noetherian valuation ring is a PID*

$[J.J.Rotman, An\ Introduction\ to\ Homological\ Algebra, page 170]$,
$[Pete\ L.Clark, Commutative\ Algebra, page 262]$

Lemma 2.6. *Let $K \subseteq L$ be an extention of fields. If $B \subseteq L$ is a valuation ring, then $A = K \cap B$ is a valuation ring.*

$[N.Bourbaki, Commutative\ algebra, page\ 379]$

Definition 2.7. *A ring R is said to be semihereditary if every finitly generated ideal of R is projective as R-module.*

By proposition 2.2 a valuation domain R is a semihereditary domain. Indeed every finitly generated ideal of R is principal and R is a domain so every finitly generated ideal of R is free as an R-module, hence R is semihereditary.

$[T.Y, Lam, Lectures\ on\ Modules\ and\ Rings\ page 45]$

There are several equivalent definitions for a Prüfer domain. We list some important ones.

Definition 2.8. *A domain R with field of quotient K is called a Prüfer domain, if every finitely generated ideal I of R is invertible; that is if $I^{-1} = \{x \in K | xI \subseteq R\}$ then $II^{-1} = R$.*

$[S.Glaz, Commutative\ Coherent\ Rings,\ page\ 24]$

Proposition 2.9. *Let $0 \neq I \triangleleft R$. Then I as R-module is projective if and only if I is invertible.*

$[D.S.Passman, A\ Course\ in\ Ring\ Theory,\ page\ 65]$

Definition 2.10. *A commutative semihereditary domain is called a Prüfer domain.*

$[T.Y, Lam, Lectures\ on\ Modules\ and\ Rings\ page 43]$

Theorem 2.11. *The following statements for an integral domain R are equivalent.*

(1) *R is a Prüfer domain.*

(2) *for every prime ideal P, R_P is a valuation ring.*
(3) *for every maximal ideal m, R_m is a valuation ring.*
(4) *Every ideal of R generated by two elements is invertible.*

[$I.Kaplansky.$ $Commutative$ $rings$ $page39$]

Example $\mathbb{Z}$ is a prüfer domain.

Theorem 2.12. *Let R be a Prüfer domain with quotient field K and Let V be a valuation ring such that $R \subseteq V \subseteq K$. Then $V = R_P$ for some prime ideal P in R*

[$I.Kaplansky.$ $Commutative$ $rings$ $page39$]

Note We know that every subring of $\mathbb{Q}$ contains $\mathbb{Z}$. So we have;

Corollary 2.13. *Every valuation ring of the field $\mathbb{Q}$ is of the form $\mathbb{Z}_{(P)}$, where P is a prime number in $\mathbb{Z}$*

[$N.Bourbaki, Commutative\ algebra, page$ 380]

Note

(1) If R is a prüfer domain, so is every localization of R and every qoutient ring of R by a prime ideal.

(2) A noetherian prüfer domain is a Dedekind domain. (i.e., R is a Dedekind domain if and only if R_m is a DVR for each maximal ideal m of R.)

[$S.Glaz, Commutative\ Coherent\ Rings,\ page$ 24, 27]

Lemma 2.14. *The nilradical of a valuation ring R is the minimal prime ideal of R.*

[$L.Fuches, L.Salce, Modules\ over\ Valuation\ Rings, Googlebook, page2$]

Lemma 2.15. *Let A be a valuation ring and b a proper ideal of A. Then $c = rad(b)$ is a prime ideal.*

[$N.Bourbaki, Commutative\ algebra, page$ 414]

Lemma 2.16. *A valuation ring R is Artinian if and only if it has a finite number of ideals.*

[$L.Fuches, L.Salce, Modules\ over\ Valuation\ Rings, Googlebook, page3$]

Lemma 2.17. *A valuation ring is noetherian if and only if it is a discrete valuation ring or a field.*

[$H.Matsumura, Commutative\ Ring\ Theory, page78$]

Example
In this example we obtaion a non-noetherian valuation ring;
Take $\mathbb{Z}^2$ with the lexicographic order . Define the valuation $v : K(x, y)^* \longrightarrow \mathbb{Z}^2$ as follows: For any $a \in K^*$ and for any $a \in K^*$ and $0 \leq n, m \in \mathbb{Z}$ set $v(ax^n y^m) = (n, m)$. For a polynomial $f = \sum f_i \in k[x, y]$ set $v(f) = inf\{v(f_0), ..., v(f_d)\}$

where the f_i are distinct monomials. Finally for a rational function $f \in k(x,y)$ there are $g,h \in k[x,y]$ such that $f = \dfrac{g}{h}$ set $v(f) = v(g) - v(h)$. The corresponding valuation ring $R_v = \{f|v(f) \geq 0\} \cup \{0\}$ contains $k[x,y]$, but it also contains xy^{-1} since $(0,0) < (1,-1)$. In fact $xy^n \in R_v$ for any n. It follows that $R_v \supseteq k[x,y,x/y,x/y^2,x/y^3...]$. So R_v is not Noetherian, hence is not a disceret valuation ring.

Question
Are valuation rings, coherent ?
For answer , we need some definitions;

Definition A module P is finitely related if there is an exact sequence of R-modules $0 \to K \to F \to P \to 0$ where F is free and K is finitely generated.

Definition An R-module P is finitely presented iff it is finitely generated and finitely related.

Definition A (left) coherent ring is a ring in which every finitely generated left ideal is finitely presented

In a valuation ring any finitely generated ideal is principal so is finitely presented: Indeed, Let I be any finitely generated ideal in valuation ring R. By 2.2 $I = Ra$ for some $a \in R$. We have $f : R \longrightarrow Ra$ is an isimorphism and R is a free R-module, there is an exact sequence $0 \to \mathbf{Ker}(f) \to R \to Ra \to 0$ that is $\mathbf{Ker}(f) = 0$ is finitely generated. Thus I is finitely presented and therefor R is a coherent ring.

3. Modules over valuation ring

Lemma 3.1. *If R is a valuation ring, then an R-module M is flat if and only if it is torsion free.*

[*H.Matsumura, Commutative Ring Theory, page*77]

Note Every finitely generated torsion free R-module over a valuation domain is free.

[*S.Glaz, Commutative Coherent Ring* , *page*24]

Corollary 3.2. *A commutative domain R is a Prüfer domain if and only if every finitly generated torsion-free R-module is projective.*

[*T.Y, Lam, Lectures on Modules and Rings page*44]

Note Every torsion-free module over a Prüfer domain is flat.

[*S.Glaz, Commutative Coherent Ring* , *page*25]

Theorem 3.3. *Over a valuation domain R, every finitely generated R-module M contains an essential pure submodule B which is a direct sum of cyclic modules. B is unique up to isomorphism.*

[L. Fuchs, L. Salce,
Modules Over non Noetherian Domains,Google book, page 170]

modules over DVRs

Lemma 3.4. *Let R be a DVR with uniformizing element t, and let $a \in \mathbb{Z}^+$. Then the ring $R_a = \frac{R}{<t^a>}$ is self- injective.*

Theorem 3.5. *Let R be a DVR with uniformizing element t, and let M be a finitely generated R-module. Then*

a) There is $N \in \mathbb{N}$ and positive integers $n, a_1 \geq a_2 \geq \ldots \geq a_n$ such that

$$M \cong R^N \oplus \left(\bigoplus_{i=1}^{n} \frac{R}{< t^{a_i} >} \right)$$

b) The numbers $N, n, a_1, a_2, \ldots, a_n$ are invariant of the isomorphism class of the module M: i.e., they are the same for any two decomposition as above.

[$Pete\ L.Clark, Commutative\ Algebra, page 271$]

Definition 3.6. *A uniserial module M is a module over a ring R, whose submodules are linearly ordered by inclusion. A ring R is called a right(left) uniserial ring if it is uniserial as a right(left) module over itself. Commutative uniserial ring is known as a valuation ring.*

properties of uniserial modules

1) For an R-module N the flowing are equivalent:
a) N is uniserial;
b)The cyclic submodules of N are linearly ordered;
c) Any submodule of N has at most one maximal submodule.
d) For any finitely generated submodule $o \neq K \subset N$, $\frac{K}{Rad(K)}$ is simple.
e)For every factor module L of N, $Soc(L)$ is simple.

2) Let N be a non-zero uniserial R-module. Then
a) Submodules and factor modules of N are again uniserial;
b) N is uniform, and finitely generated submodule of N are cyclic;
c) If N is noetherian, there exists a possibly finite descending chain of submodules $N = N_1 \supset N_2 \supset \ldots$ with simple factores $\frac{N_i}{N_{i+1}}$;
d) If N is artinian, there exists a possibly finite ascending chain of submodules $0 = S_0 \subset S_1 \subset S_2 \subset \ldots$ with simple factors $\frac{S_{i+1}}{S_i}$;
e) If N has finite length, there is a unique composition series.

[$R.Wisbauer, Foundation\ of\ Module\ and\ Ring\ page 539$]

Example

(1) The valuatiom domaine R itself is a uniserial R- module.

(2) Examples of uniserial $\mathbb{Z}$-modules are the modules $\frac{\mathbb{Z}}{p^k\mathbb{Z}}$ for any $k, p \in \mathbb{N}$, p a prime number. They have the unique composition series

$$\frac{\mathbb{Z}}{p^k\mathbb{Z}} \supset \frac{p\mathbb{Z}}{p^k\mathbb{Z}} \supset \ldots \supset \frac{p^{k-1}\mathbb{Z}}{p^k\mathbb{Z}} \supset 0.$$

(3) If R is a valuation ring, then its field of fraction is a uniserial R-module.